Chapter 7 Ratio and Proportion Ex 7.2

Question 1.
Find the value of x in the following proportions:
(i) 10 : 35 = x : 42
(ii) 3 : x = 24 : 2
(iii) 2.5 : 1.5 = x : 3
(iv) x : 50 :: 3 : 2
Solution:

(i) 10 : 35 = x : 42

$\Rightarrow 35 \times x = 10 \times 42$

$$\therefore \quad x = \frac{10 \times 42}{35} = 2 \times 6 = 12$$

(ii) $3 : x = 24 : 2$

$\Rightarrow x \times 24 = 3 \times 2$

$$\therefore \quad x = \frac{3 \times 2}{24} = \frac{1}{4}$$

(iii) $2{\cdot}5 : 1{\cdot}5 = x : 3$

$\Rightarrow 1{\cdot}5 \times x = 2{\cdot}5 \times 3$

$$x = \frac{2{\cdot}5 \times 3}{1{\cdot}5} = 5{\cdot}0$$

(iv) $x : 50 : : 3 : 2$

$\Rightarrow x \times 2 = 50 \times 3$

$$x = \frac{50 \times 3}{2} = 75$$

Question 2.
Find the fourth proportional to
(i) 3, 12, 15
(ii) 13,14,15
(iii) 1.5, 2.5, 4.5
(iv) 9.6 kg, 7.2 kg, 28.8 kg
Solution:

(i) Let fourth proportional to

3, 12, 15 be x.

then 3 : 12 :: 15 : x

$$\Rightarrow 3 \times x = 12 \times 15$$

$$x = \frac{12 \times 15}{3} = 60$$

(*ii*) Let fourth proportional to $\frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ be x

then $\frac{1}{3} : \frac{1}{4} :: \frac{1}{5} : x$

$$\Rightarrow \frac{1}{3} \times x = \frac{1}{4} \times \frac{1}{5}$$

$$\Rightarrow x = \frac{1}{4} \times \frac{1}{5} \times \frac{3}{1} = \frac{3}{20}$$

(*iii*) Let fourth proportional to

1·5, 2·5, 4·5 be x

then 1·5 : 2·5 : : 4·5 : x

$\therefore$ 1·5 × x = 2·5 × 4·5

$$x = \frac{2 \cdot 5 \times 4 \cdot 5}{1 \cdot 5} = 7 \cdot 5$$

(*iv*) Let fourth proportional to 9·6 kg, 7·2 kg, 28·8 kg be x

then 9·6 : 7·2 : : 28·8 : x

$$\Rightarrow 9 \cdot 6 \times x = 7 \cdot 2 \times 28 \cdot 8$$

$$x = \frac{7 \cdot 2 \times 28 \cdot 8}{9 \cdot 6} = 21 \cdot 6$$

Question 3.
Find the third proportional to
(i) 5, 10
(ii) 0.24, 0.6
(iii) Rs. 3, Rs. 12

(iv) 514 and 7.

Solution:

(i) Let x be the third proportional to 5, 10,

then 5 : 10 :: 10 : x

$$\therefore\ 5 \times x = 10 \times 10 \Rightarrow x = \frac{10 \times 10}{5} = 20$$

$\therefore$ Third proportional = 20

(ii) Let x be the third proportional to 0·24, 0·6

then 0·24 : 0·6 : : 0·6 : x

$\therefore\ 0{\cdot}24 \times x = 0{\cdot}6 \times 0{\cdot}6$

$$x = \frac{0{\cdot}6 \times 0{\cdot}6}{0{\cdot}24} = 1{\cdot}5$$

$\therefore$ Third proportional = 1·5

(iii) Let x be the third proportional to Rs. 3 and Rs. 12

then Rs. 3 : Rs. 12 : : Rs. 12 : x

$$\therefore\ x = \frac{12 \times 12}{3} = 48$$

$\therefore$ Third proportional = Rs. 48

(iv) Let x be the third proportional to $5\frac{1}{4}$ and 7

$$\text{then } 5\frac{1}{4} : 7 :: 7 : x \Rightarrow \frac{21}{4} : 7 :: 7 : x$$

$$\therefore\ \frac{21}{4} \times x = 7 \times 7$$

$$x = \frac{7 \times 7 \times 4}{21} = \frac{28}{3} = 9\frac{1}{3}$$

$\therefore$ Third proportional $= 9\frac{1}{3}$

Question 4.

Find the mean proportion of:

(i) 5 and 80

(ii) 112 and 175

(iii) 8.1 and 2.5

(iv) (a – b) and ($a^3 - a^2b$), a> b

Solution:

(i) Let x be the mean proportion of 5 and 80 ,

then 5 : x : : x : 80

$x^2 = 5 \times 80$

$\Rightarrow x = \sqrt{5 \times 80} = \sqrt{400} = 20$

x = 20

Hence mean proportion = 20

(*ii*) Let x be the mean proportion of $\frac{1}{12}$ and $\frac{1}{75}$

then $\frac{1}{12} : x :: x : \frac{1}{75}$

$\therefore x^2 = \frac{1}{12} \times \frac{1}{75} = \frac{1}{900}$

$\therefore \quad x = \sqrt{\frac{1}{900}} = \frac{1}{30}$

Hence the mean proportion $= \frac{1}{30}$

(*iii*) Let the x be the mean proportion of 8·1 and 2·5

$\therefore \quad 8{\cdot}1 : x : : x : 2{\cdot}5$

$\therefore \quad x^2 = 8{\cdot}1 \times 2{\cdot}5$

$\therefore \quad x = \sqrt{8{\cdot}1 \times 2{\cdot}5} = \sqrt{20{\cdot}25} = 4{\cdot}5$

Hence mean proportion = 4·5

(*iv*) Let x be the mean proportion to $(a - b)$ and $(a^3 - a^2 b)$, $a > b$

then $(a - b) : x : : x : (a^3 - a^2 b)$

$x^2 = (a - b)(a^3 - a^2 b)$

$= (a - b)\, a^2 (a - b) = a^2 (a - b)^2$

$\therefore x = a(a - b)$

Hence the mean proportion $= a(a - b)$

Question 5.

If a, 12, 16 and b are in continued proportion find a and b.

Solution:

∵ a, 12, 16, b are in continued proportion, then

$$\frac{a}{12}=\frac{12}{16}=\frac{16}{b} \Rightarrow \frac{a}{12}=\frac{12}{16} \Rightarrow 16a=144$$

$$\Rightarrow a=\frac{144}{16}=9$$

$$\text{and } \frac{12}{16}=\frac{16}{b} \Rightarrow 12b=16\times16=256$$

$$b=\frac{256}{12}=\frac{64}{3}=21\frac{1}{3}$$

Hence $a=9$, $b=\frac{64}{3}$ or $21\frac{1}{3}$

Question 6.

What number must be added to each of the numbers 5, 11, 19 and 37 so that they are in proportion? (2009)

Solution:

Let x be added to 5, 11, 19 and 37 to make them in proportion.

5 + x : 11 + x : : 19 + x : 37 + x

$$\Rightarrow (5+x)(37+x)=(11+x)(19+x)$$

$$\Rightarrow 185+5x+37x+x^2=209+11x+19x+x^2$$

$$\Rightarrow 185+42x+x^2=209+30x+x^2$$

$$\Rightarrow 42x-30x+x^2-x^2=209-185$$

$$\Rightarrow 12x=24 \Rightarrow x=2$$

∴ Least number to be added = 2

Question 7.

What number should be subtracted from each of the numbers 23, 30, 57 and 78 so that the remainders are in proportion? (2004)

Solution:

Let x be subtracted from each term, then

23 − x, 30 − x, 57 − x and 78 − x are proportional

23 − x : 30 − x : : 57 − x : 78 − x

$$\Rightarrow \frac{23-x}{30-x} = \frac{57-x}{78-x}$$

$$\Rightarrow (23-x)(78-x) = (30-x)(57-x)$$

$$\Rightarrow 1794 - 23x - 78x + x^2$$

$$= 1710 - 30x - 57x + x^2$$

$$\Rightarrow x^2 - 101x + 1794 = x^2 - 87x + 1710$$

$$\Rightarrow x^2 - 101x + 1794 - x^2 + 87x - 1710 = 0$$

$$\Rightarrow -14x + 84 = 0 \Rightarrow 14x = 84$$

$$\therefore x = \frac{84}{14} = 6$$

Hence 6 is to be subtracted

Question 8.

If 2x – 1, 5x – 6, 6x + 2 and 15x – 9 are in proportion, find the value of x.

Solution:

∵ 2x − 1, 5x − 6, 6x + 2 and 15x − 9 are in proportion.

then (2x − 1) (15x − 9) = (5x − 6) (6x + 2)

$$\Rightarrow 30x^2 - 18x - 15x + 9 \Rightarrow 30x^2 + 10x - 36x - 12$$

$$\Rightarrow 30x^2 - 33x + 9 = 30x^2 - 26x - 12$$

$$\Rightarrow 30x^2 - 33x - 30x^2 + 26x = -12 - 9$$

$$\Rightarrow -7:: = -21$$

$$\therefore \quad x = \frac{-21}{-7} = 3$$

Hence $x = 3$

Question 9.

If x + 5 is the mean proportion between x + 2 and x + 9, find the value of x.

Solution:

$\because$ x + 5 is the mean proportion between x + 2 and x + 9, then

$(x + 5)^2 = (x + 2)(x + 9)$

$\Rightarrow x^2 + 10x + 25 = x^2 + 11x + 18$

$\Rightarrow x^2 + 10x - x^2 - 11x = 18 - 25$

$\Rightarrow -x = -7$

$\because x = 7$

Question 10.
What number must be added to each of the numbers 16, 26 and 40 so that the resulting numbers may be in continued proportion?
Solution:

Let x be added to each number then

16 + x, 26 + x and 40 + x

are in continued proportion.

$$\Rightarrow \frac{16+x}{26+x} = \frac{26+x}{40+x}$$

Cross Multiplying,

$$(16 + x)(40 + x) = (26 + x)(26 + x)$$

$$\Rightarrow 640 + 16x + 40x + x^2 = 676 + 26x + 26x + x^2$$

$$\Rightarrow 640 + 56x + x^2 = 676 + 52x + x^2$$

$$\Rightarrow 56x + x^2 - 52x - x^2 = 676 - 640$$

$$\Rightarrow 4x = 36 \Rightarrow x = \frac{36}{4} = 9$$

$\therefore$ 9 is to be added.

Question 11.
Find two numbers such that the mean proportional between them is 28 and the third proportional to them is 224.

Solution:

Let the two numbers are a and b.

∵ 28 is the mean proportional

∵ a : 28 : : 28 : b

$\therefore\ ab = (28)^2 = 784 \Rightarrow\ a = \frac{784}{b}$ ...(i)

∵ 224 is the third proportional

$\therefore\ a : b : : b : 224$

$\Rightarrow\ b^2 = 224a$ ...(ii)

Substituting the value of a in (ii)

$$b^2 = 224 \times \frac{784}{b} \Rightarrow b^3 = 224 \times 784$$

$$\Rightarrow\ b^3 = 175616 = (56)^3$$

$\therefore\ b = 56$

Now substituting the value of b in (i)

$$a = \frac{784}{56} = 14$$

Hence numbers are 14, 56

Question 12.

If b is the mean proportional between a and c, prove that a, c, $a^2 + b^2$, and $b^2 + c^2$ are proportional.

Solution:

∵ b is the mean proportional between a and c, then,

$b^2 = a \times c \Rightarrow b^2 = ac$ ...(i)

Now $a, c, a^2 + b^2$ and $b^2 + c^2$ are in proportion

if $\frac{a}{c} = \frac{a^2 + b^2}{b^2 + c^2}$

if $a(b^2 + c^2) = c(a^2 + b^2)$

if $a(ac + c^2) = c(a^2 + ac)$ [from (i)]

if $ac(a + c) = a^2c + ac^2$

if $ac(a + c) = ac(a + c)$ which is true.

Hence proved.

Question 13.
If b is the mean proportional between a and c, prove that (ab + bc) is the mean proportional between $(a^2 + b^2)$ and $(b^2 + c^2)$.
Solution:

b is the mean proportional between a and c then

$b^2 = ac$ ...(i)

Now if (ab + bc) is the mean proportional

between $(a^2 + b^2)$ and $(b^2 + c^2)$, then

$(ab + bc)^2 = (a^2 + b^2)(b^2 + c^2)$

Now L.H.S. $= (ab + bc)^2 = a^2b^2 + b^2c^2 + 2ab^2c$

$= a^2(ac) + ac(c)^2 + 2a\,ac.\,c$ [from (i)]

$= a^3c + ac^3 + 2a^2c^2$

$= ac(a^2 + c^2 + 2ac) = ac(a + c)^2$

R.H.S. $= (a^2 + b^2)(b^2 + c^2)$

$= (a^2 + ac)(ac + c^2)$ [from (i)]

$= a(a + c)\,c(a + c) = ac(a + c)^2$

$\therefore$ L.H.S. = R.H.S.

Question 14.
If y is mean proportional between x and z, prove that
$xyz\,(x + y + z)^3 = (xy + yz + zx)^3$.
Solution:

$\because$ y is the mean proportional between

x and z, then

$y^2 = xz$ ...(i)

L.H.S. $= xyz\,(x + y + z)^3$

$= xz,\,y\,(x + y + z)^3$

$= y^2\,y\,(x + y + z)^3$ [from (i)]

$= y^3(x + y + z)^3 \quad = [y(x + y + z)]^3$

$= [xy + y^2 + yz]^3 = (xy + yz + zx)^3$ (from i)

= R.H.S.

Question 15.
If a + c = mb and 1b+1d=mc, prove that a, b, c and d are in proportion.

Solution:

a + c = mb and $\frac{1}{b} + \frac{1}{d} = \frac{m}{c}$

a + c = mb

$$\frac{a}{b} + \frac{c}{d} = m \qquad \text{(Dividing by } b\text{) ...(}i\text{)}$$

and $\frac{1}{b} + \frac{1}{d} = \frac{m}{c}$

$$\frac{c}{b} + \frac{c}{d} = m \qquad \text{(Multiplying by } c\text{) ...(}ii\text{)}$$

From (*i*) and (*ii*),

$$\frac{a}{b} + \frac{c}{b} = \frac{c}{b} + \frac{c}{d} \Rightarrow \frac{a}{b} = \frac{c}{d}$$

Hence, *a*, *b*, *c* and *d* are proportional.

Question 16.

If $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$, prove that

(i) $\frac{x^3}{a^2} + \frac{y^3}{b^2} + \frac{z^3}{c^2} = \frac{(x+y+z)^3}{(a+b+c)^2}$

(*ii*) $\left[\frac{a^2x^2 + b^2y^2 + c^2z^2}{a^3x + b^3y + c^3z}\right]^3 = \frac{xyz}{abc}$

(*iii*) $\frac{ax - by}{(a+b)(x-y)} + \frac{by - cz}{(b+c)(y-z)} + \frac{cz - ax}{(c+a)(z-x)} = 3$

Solution:

$\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$

∴ x = ak, y = bk, z = ck

(i) L.H.S. $= \frac{x^3}{a^2} + \frac{y^3}{b^2} + \frac{z^3}{c^2}$

$= \frac{a^3k^3}{a^2} + \frac{b^3k^3}{b^2} + \frac{c^3k^3}{c^2}$

$= a k^3 + bk^3 + ck^3 = k^3 (a + b + c)$

R.H.S. $= \frac{(x + y + z)^3}{(a + b + c)^2}$

$= \frac{(ak + bk + ck)^3}{(a + b + c)^2} = \frac{k^3 (a + b + c)^3}{(a + b + c)^2}$

$= k^3 (a + b + c)$

Hence L.H.S. = R.H.S.

(ii) L.H.S $= \left[\frac{a^2x^2 + b^2y^2 + c^2z^2}{a^3x + b^3y + c^3z}\right]^3$

$= \left[\frac{a^2.a^2k^2 + b^2.b^2k^2 + c^2.c^2k^2}{a^3.a.k + b^3.bk + c^3.ck}\right]^3$

$$= \left[\frac{a^4k^2 + b^4k^2 + c^4k^2}{a^4k + b^4k + c^4k} \right]^3$$

$$= \left[\frac{k^2(a^4 + b^4 + c^4)}{k(a^4 + b^4 + c^4)} \right]^3 = k^3$$

$$\text{R.H.S} = \frac{xyz}{abc} = \frac{ak.bk.ck}{abc} = k^3$$

$\therefore$ L.H.S = R.H.S

(*iii*) L.H.S $\dfrac{ax - by}{(a+b)(x-y)} + \dfrac{by - cz}{(b+c)(y-z)}$

$$+ \frac{cz - ax}{(c+a)(z-x)}$$

$$= \frac{a.ak - b.bk}{(a+b)(ak-bk)} + \frac{b.bk - c.ck}{(b+c)(bk-ck)}$$

$$+ \frac{c.ck - a.ak}{(c+a)(ck-ak)}$$

$$= \frac{a^2k - b^2k}{(a+b)k(a-b)} + \frac{b^2k - c^2k}{(b+c)k(b-c)}$$

$$+ \frac{c^2k - a^2k}{(c+a)k(c-a)}$$

$$= \frac{k(a^2 - b^2)}{k(a^2 - b^2)} + \frac{k(b^2 - c^2)}{k(b^2 - c^2)} + \frac{k(c^2 - a^2)}{k(c^2 - a^2)}$$

$$= 1 + 1 + 1 = 3 = \text{R.H.S}$$

Question 17.

If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$ prove that :

(i) $(b^2 + d^2 + f^2)(a^2 + c^2 + e^2) = (ab + cd + ef)^2$

$$(ii)\quad \frac{(a^3+c^3)^2}{(b^3+d^3)^2} = \frac{e^6}{f^6}$$

$$(iii)\quad \frac{a^2}{b^2} + \frac{c^2}{d^2} + \frac{e^2}{f^2} = \frac{ac}{bd} + \frac{ce}{df} + \frac{ae}{df}$$

$$(iv)\quad bdf\left[\frac{a+b}{b} + \frac{c+d}{d} + \frac{c+f}{f}\right]^3 = 27\,(a+b)(c+d)(e+f)$$

Solution:

$\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$ = k(say)

$\therefore$ a = bk, c = dk, e = fk

$$\begin{aligned}
(i)\ \text{L.H.S.} &= (b^2 + d^2 + f^2)(a^2 + c^2 + e^2) \\
&= (b^2 + d^2 + f^2)(b^2k^2 + d^2k^2 + f^2k^2) \\
&= (b^2 + d^2 + f^2)k^2(b^2 + d^2 + f^2) \\
&= k^2(b^2 + d^2 + f^2) \\
\text{R.H.S} &= (ab + cd + ef)^2 \\
&= (b.\ kb. + dk.\ d + fk.\ f)^2 \\
&= (kb^2 + kd^2 + kf^2) = k^2(b^2 + d^2 + f^2)^2
\end{aligned}$$

$\therefore$ L.H.S = R.H.S

$$(ii)\ \text{L.H.S}\quad \frac{(a^3+c^3)^2}{(b^3+d^3)^2} = \frac{(b^3k^3+d^3k^3)^2}{(b^3+d^3)^2}$$

$$= \frac{[k^3(b^3+d^3)]^2}{(b^3+a^3)^2} = \frac{k^6(b^3+d^3)^2}{(b^3+d^3)^2} = k^6$$

$$\text{R.H.S} = \frac{e^6}{f^6} = \frac{f^6k^6}{f^6} = k^6$$

$\therefore$ L.H.S = R.H.S

(iii) L.H.S $= \frac{a^2}{b^2} + \frac{c^2}{d^2} + \frac{e^2}{f^2} = \frac{b^2k^2}{b^2}$

$+ \frac{d^2k^2}{d^2} + \frac{f^2k^2}{f^2} = k^2 + k^2 + k^2 = 3k^2$

R.H.S $= \frac{ac}{bd} + \frac{ce}{df} + \frac{ae}{bf}$

$= \frac{bk.dk}{b.d} + \frac{dk.fk}{d.f} + \frac{bk.fk}{b.f}$

$= k^2 + k^2 + k^2 = 3k^2$

$\therefore$ L.H.S = R.H.S

(iv) L.H.S $= bdf\left[\frac{a+b}{b} + \frac{c+d}{d} + \frac{e+f}{f}\right]^3$

$= bdf\left[\frac{bk+b}{b} + \frac{dk+d}{d} + \frac{fk+f}{f}\right]^3$

$= bdf\left[\frac{b(k+1)}{b} + \frac{d(k+1)}{d} + \frac{f(k+1)}{f}\right]^3$

$= bdf(k+1+k+1+k+1)^3$

$= bdf(3k+3)^3 = 27\,bdf(k+1)^3$

R.H.S $= 27(a+b)(c+d)(e+f)$

$= 27(bk+b)(dk+d)(fk+f)$

$= 27\,b(k+1)\,d(k+1)\,f(k+1)$

$= 27\,bdf(k+1)^3$

$\therefore$ L.H.S = R.H.S

Question 18.
If ax = by = cz; prove that
$\frac{x^2}{yz} + \frac{y^2}{zx} + \frac{z^2}{xy} = \frac{bc}{a^2} + \frac{ca}{b^2} + \frac{ab}{c^2}$

Solution:

Let ax = by = cz = k

$$\therefore \quad x = \frac{k}{a}, y = \frac{k}{b}, z = \frac{k}{c}$$

$$\text{L.H.S.} = \frac{x^2}{yz} + \frac{y^2}{zx} + \frac{z^2}{xy}$$

$$= \frac{\frac{k^2}{a^2}}{\frac{k}{b}\cdot\frac{k}{c}} + \frac{\frac{k^2}{b^2}}{\frac{k}{c}\cdot\frac{k}{a}} + \frac{\frac{k^2}{c^2}}{\frac{k}{a}\cdot\frac{k}{b}} = \frac{\frac{k^2}{a^2}}{\frac{k^2}{bc}} + \frac{\frac{k^2}{b^2}}{\frac{k^2}{ca}} + \frac{\frac{k^2}{c^2}}{\frac{k^2}{ab}}$$

$$= \frac{k^2}{a^2} \times \frac{bc}{k^2} + \frac{k^2}{b^2} \times \frac{ca}{k^2} + \frac{k^2}{c^2} \times \frac{ab}{k^2}$$

$$= \frac{bc}{a^2} + \frac{ca}{b^2} + \frac{ab}{c^2} = \text{R.H.S}$$

Question 19.

If a, b, c and d are in proportion, prove that:

(i) (5a + 7b) (2c – 3d) = (5c + 7d) (2a – 3b)

(ii) (ma + nb) : b = (mc + nd) : d

(iii) $(a^4 + c^4) : (b^4 + d^4) = a^2 c^2 : b^2 d^2$.

$$(iv) \quad \frac{a^2 + ab}{c^2 + cd} = \frac{b^2 - 2ab}{d^2 - 2cd}$$

$$(v) \quad \frac{(a+c)^3}{(b+d)^3} = \frac{a(a-c)^2}{b(b-d)^2}$$

$$(vi) \quad \frac{a^2 + ab + b^2}{a^2 - ab + b^2} = \frac{c^2 + cd + d^2}{c^2 - cd + d^2}$$

$$(vii) \quad \frac{a^2 + b^2}{c^2 + d^2} = \frac{ab + ad - bc}{bc + cd - ad}$$

$$(viii) \ abcd \left[\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2}\right] = a^2 + b^2 + c^2 + d^2$$

Solution:

∵ a, b, c, d are in proportion

$\frac{a}{b} = \frac{c}{d} = k$(say)

$a = bk,\ c = dk.$

(i) L.H.S. $= (5a + 7b)(2c - 3d)$

$= (5 \cdot bk + 7b)(2\,dk - 3d)$

$= k(5b + 7b)\,k(2d - 3d)$

$= k^2 (12b) \times (-d) = -12\,bd\,k^2$

R.H.S. $= (5c + 7d)(2a - 3b)$

$= (5dk + 7d)(2\,k \cdot b - 3b)$

$= k(5d + 7d)\,k(2b - 3b)$

$= k^2 (12d)(-b) = -12k^2\,bd = -12\,bd\,k^2$

∴ L.H.S = R.H.S

(ii) $(ma + nb) : b = (mc + nd) : d$

$$\Rightarrow \frac{ma + nb}{b} = \frac{mc + nd}{d}$$

$$\text{L.H.S.} = \frac{mbk + nb}{b} = \frac{b(mk + n)}{b}$$

$= mk + n$

$$\text{R.H.S.} = \frac{mc + nd}{d} = \frac{mdk + nd}{d}$$

$$= \frac{d(mk + n)}{d} = mk + n.$$

∴ L.H.S = R.H.S.

(*iii*) $(a^4 + c^4) : (b^4 + d^4) = a^2 c^2 : b^2 d^2$

$$\frac{a^4 + c^4}{b^4 + d^4} = \frac{a^2 c^2}{b^2 d^2}$$

$$\text{L.H.S.} = \frac{a^4 + c^4}{b^4 + d^4} = \frac{b^4 k^4 + d^4 k^4}{b^4 + d^4}$$

$$= \frac{k^4 (b^4 + d^4)}{(b^4 + d^4)} = k^4$$

$$\text{R.H.S.} = \frac{a^2 c^2}{b^2 d^2} = \frac{k^2 b^2 . k^2 d^2}{b^2 . d^2} = k^4$$

Hence L.H.S. = R.H.S.

(*iv*) $\text{L.H.S} = \frac{a^2 + ab}{c^2 + cd} = \frac{k^2 b^2 + bk.b}{k^2 d^2 + dk.d}$

$$= \frac{kb^2 (k+1)}{d^2 k (k+1)} = \frac{b^2}{d^2}$$

$$\text{R.H.S} = \frac{b^2 - 2ab}{d^2 - 2cd} = \frac{b^2 - 2.bkb}{d^2 - 2dkd}$$

$$= \frac{b^2 (1-2k)}{d^2 (1-2k)} = \frac{b^2}{d^2}$$

$\therefore$ L.H.S = R.H.S

(v) L.H.S. $= \dfrac{(a+c)^3}{(b+d)^3} = \dfrac{(bk+dk)^3}{(b+d)^3}$

$= \dfrac{k^3(b+d)^3}{(b+d)^3} = k^3$

R.H.S. $= \dfrac{a(a-c)^2}{b(b-d)^2} = \dfrac{bk(bk-dk)^2}{b(b-d)^2}$

$= \dfrac{bk.k^2(b-d)^2}{b(b-d)^2} = k^3$

$\therefore$ L.H.S = R.H.S.

(vi) L.H.S $= \dfrac{a^2+ab+b^2}{a^2-ab+b^2}$

$= \dfrac{b^2k^2+bk.b+b^2}{b^2k^2-bk.b+b^2}$

$= \dfrac{b^2(k^2+k+1)}{b^2(k^2-k+1)} = \dfrac{k^2+k+1}{k^2-k+1}$

R.H.S $= \dfrac{c^2+cd+d^2}{c^2-cd+d^2}$

$= \dfrac{d^2k^2+dkd+d^2}{d^2k^2-dk.d+d^2}$

$$= \frac{d^2(k^2+k+1)}{d^2(k^2-k+1)} = \frac{k^2+k+1}{k^2-k+1}$$

$\therefore$ L.H.S = R.H.S

(vii) $$\text{L.H.S} = \frac{a^2+b^2}{c^2+d^2} = \frac{b^2k^2+b^2}{d^2k^2+d^2}$$

$$= \frac{b^2(k^2+1)}{d^2(k^2+1)} = \frac{b^2}{d^2}$$

$$\text{R.H.S} = \frac{ab+ad-bc}{bc+cd-ad}$$

$$= \frac{bk.b+bk.d-b.dk}{b.kd+dk.d-bk.d}$$

$$= \frac{k(b^2+bd-bd)}{k(bd+d^2-bd)} = \frac{b^2}{d^2}$$

$\therefore$ L.H.S = R.H.S.

(viii) $$\text{L.H.S.} = abcd\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}\right)$$

$$= bk.b.\ dk.d\left[\frac{1}{b^2k^2}+\frac{1}{b^2}+\frac{1}{d^2k^2}+\frac{1}{d^2}\right]$$

$$= k^2b^2d^2\left[\frac{d^2+d^2k^2+b^2+b^2k^2}{b^2d^2k^2}\right]$$

$$= d^2(1+k^2)+b^2(1+k^2) = (1+k^2)(b^2+d^2)$$

$$\text{R.H.S} = a^2+b^2+c^2+d^2$$

$$= b^2k^2+b^2+d^2k^2+d^2$$

$$= b^2(k^2+1)+d^2(k^2+1) = (k^2+1)(b^2+d^2)$$

$\therefore$ L.H.S = R.H.S.

Question 20.

If x, y, z are in continued proportion, prove that: $\frac{(x+y)^2}{(y+z)^2}=\frac{x}{z}$. (2010)

Solution:

x, y, z are in continued proportion

Let $\frac{x}{y} = \frac{y}{z} = k$

Then $y = kz$

$x = yk = kz \times k = k^2 z$

Now L.H.S. $= \frac{(x+y)^2}{(y+z)^2}$

$= \frac{(k^2 z + kz)^2}{(kz+z)^2} = \frac{\{kz(k+1)\}^2}{\{z(k+1)\}^2}$

$= \frac{k^2 z^2 (k+1)^2}{z^2 (k+1)^2} = k^2$

R.H.S. $= \frac{x}{z} = \frac{k^2 z}{z} = k^3$

$\therefore$ L.H.S. = R.H.S.

Question 21.

If a, b, c are in continued proportion, prove that:

$pa^2+qab+rb^2pb^2+qbc+rc^2=ac$

Solution:

Given a, b, c are in continued proportion

$\frac{pa^2+qab+rb^2}{pb^2+qbc+rc^2} = \frac{a}{c}$

Let $\frac{a}{b} = \frac{b}{c} = k$

$\Rightarrow\ a = bk$ and $b = ck$ ...(*i*)

$\Rightarrow\ a = (ck)k = ck^2$ [Using (*i*)]

and $b = ck$

$$\text{L.H.S.} = \frac{a}{c} = \frac{ck^2}{c} = k^2$$

$$\text{R.H.S.} = \frac{p(ck^2)^2 + q(ck^2)ck + r(ck)^2}{p(ck)^2 + q(ck)c + rc^2}$$

$$= \frac{pc^2k^4 + qc^2k^3 + rc^2k^2}{pc^2k^2 + qc^2k + rc^2}$$

$$= \frac{c^2k^2}{c^2}\left[\frac{pk^2+qk+r}{pk^2+qk+r}\right] = k^2 \quad \text{...(iii)}$$

From (*ii*) and (*iii*), L.H.S. = R.H.S.

$\therefore\ b = ck,\ a = bk = c\,k\,k = ck^2$

(*i*) L.H.S

$$= \frac{a+b}{b+c} = \frac{ck^2+ck}{ck+c} = \frac{ck(k+1)}{c(k+1)} = k$$

$$\text{R.H.S} = \frac{a^2(b-c)}{b^2(a-b)}$$

$$= \frac{(ck^2)^2(ck-c)}{(ck)^2(ck^2-ck)}$$

$$= \frac{c^2k^4\,c(k-1)}{c^2k^2\,ck(k-1)}$$

Question 22.

If a, b, c are in continued proportion, prove that:

(i) $\frac{a+b}{b+c} = \frac{a^2(b-c)}{b^2(a-b)}$

(*ii*) $\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} = \frac{a}{b^2c^2} + \frac{b}{c^2a^2} + \frac{c}{a^2b^2}$

(*iii*) $a : c = (a^2 + b^2) : (b^2 + c^2)$

(*iv*) $a^2 b^2 c^2 (a^{-4} + b^{-4} + c^{-4}) = b^{-2} (a^4 + b^4 + c^4)$

(*v*) $abc (a + b + c)^3 = (ab + bc + ca)^3$

(*vi*) $(a + b + c) (a - b + c) = a^2 + b^2 + c^2$

Solution:

As a, b, c, are in continued proportion

Let $\frac{a}{b} = \frac{b}{c} = k$

$$= \frac{c^3k^4(k-1)}{c^3k^3(k-1)} = k$$

$\therefore$ L.H.S = R.H.S

(ii) $$\text{L.H.S.} = \frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3}$$

$$= \frac{1}{(ck^2)^3} + \frac{1}{(ck)^3} + \frac{1}{c^3}$$

$$= \frac{1}{c^3k^6} + \frac{1}{c^3k^3} + \frac{1}{c^3}$$

$$= \frac{1}{c^3}\left[\frac{1}{k^6} + \frac{1}{k^3} + \frac{1}{1}\right]$$

$$\text{R.H.S.} = \frac{a}{b^2c^2} + \frac{b}{c^2a^2} + \frac{c}{a^2b^2}$$

$$= \frac{ck^2}{(ck)^2c^2} + \frac{ck}{c^2(ck^2)^2} + \frac{c}{(ck^2)^2(ck)^2}$$

$$= \frac{ck^2}{c^4k^2} + \frac{ck}{c^4k^4} + \frac{c}{c^4k^6}$$

$$= \frac{1}{c^3} + \frac{1}{c^3k^3} + \frac{1}{c^3k^6}$$

$$= \frac{1}{c^3}\left[1 + \frac{1}{k^3} + \frac{1}{k^6}\right]$$

$$= \frac{1}{c^3}\left[\frac{1}{k^6} + \frac{1}{k^3} + 1\right]$$

$\therefore$ L.H.S = R.H.S.

(*iii*) $a : c = (a^2 + b^2) : (b^2 + c^2)$

$$\Rightarrow \frac{a}{c} = \frac{a^2 + b^2}{b^2 + c^2}$$

L.H.S. $\frac{a}{c} = \frac{ck^2}{c} = k^2$

R.H.S. $= \frac{(ck^2)^2 + (ck)^2}{(ck)^2 + c^2}$

$$= \frac{c^2k^4 + c^2k^2}{c^2k^2 + c^2}$$

$$= \frac{c^2k^2(k^2 + 1)}{c^2(k^2 + 1)} = k^2$$

$\therefore$ L.H.S. = R.H.S.

(*iv*) L.H.S. $= a^2\, b^2\, c^2\, (a^{-4} + b^{-4} + c^{-4})$

$$= a^2b^2c^2\left[\frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}\right]$$

$$= \frac{a^2b^2c^2}{a^4}+\frac{a^2b^2c^2}{b^4}+\frac{a^2b^2c^2}{c^4}$$

$$= \frac{b^2c^2}{a^2}+\frac{c^2a^2}{b^2}+\frac{a^2b^2}{c^2}$$

$$= \frac{(ck)^2.c^2}{(ck^2)^2}+\frac{c^2(ck^2)^2}{(ck)^2}+\frac{(ck^2)^2\,(ck)^2}{c^2}$$

$$= \frac{c^2k^2.c^2}{c^2k^4}+\frac{c^2.c^2k^4}{c^2k^2}+\frac{c^2k^4.c^2k^2}{c^2}$$

$$= \frac{c^2}{k^2}+\frac{c^2k^2}{1}+\frac{c^2k^6}{1}$$

$$= c^2\left[\frac{1}{k^2}+k^2+k^6\right]$$

$$= \frac{c^2}{k^2}[1+k^4+k^8]$$

R.H.S. $= b^{-2}\,[a^4 + b^4 + c^4]$

$= \frac{1}{b^2}[a^4 + b^4 + c^4]$

$= \frac{1}{(ck)^2}[(ck^2)^4 + (ck)^4 + c^4]$

$= \frac{1}{c^2k^2}[c^4k^8 + c^4k^4 + c^4]$

$= \frac{c^4}{c^2k^2}[k^8 + k^4 + 1]$

$= \frac{c^2}{k^2}[1 + k^4 + k^8]$

$\therefore$ L.H.S. = R.H.S.

(*v*) L.H.S. $= abc\,(a + b + c)^3$

$= ck^2.ck.c\,[ck^2 + ck + c]^3$

$= c^3k^3\,[c\,(k^2 + k + 1)]^3$

$= c^3k^3.c^3.\,(k^2 + k + 1)^3 = c^6k^3\,(k^2 + k + 1)^3$

R.H.S. $= (ab + bc + ca)^3$

$= (ck^2\,.\,ck + ck\,.\,c + c\,.\,ck^2)^3$

$= (c^2k^3 + c^2k + c^2k^2)^3 = (c^2k^3 + c^2k^2 + c^2k)^3$

$= [c^2k\,(k^2 + k + 1)]^3 = c^6k^3\,(k^2 + k + 1)^3$

$\therefore$ L.H.S. = R.H.S.

(*vi*) L.H.S. $= (a + b + c)\,(a - b + c)$

$= (ck^2 + ck + c)\,(ck^2 - ck + c)$

$= c\,(k^2 + k + 1)\,c\,(k^2 - k + 1)$

$= c^2\,(k^2 + k + 1)\,(k^2 - k + 1)$

$= c^2\,(k^4 + k^2 + 1)$

R.H.S. $= a^2 + b^2 + c^2$

$= (ck^2)^2 + (ck)^2 + (c)^2$

$= c^2k^4 + c^2k^2 + c^2 = c^2\,(k^4 + k^2 + 1)$

$\therefore$ L.H.S. = R.H.S.

Question 23.

If a, b, c, d are in continued proportion, prove that:

(i) $\frac{a^3+b^3+c^3}{b^3+c^3+d^3}=\frac{a}{d}$

(ii) $(a^2-b^2)(c^2-d^2)=(b^2-c^2)^2$

(iii) $(a+d)(b+c)-(a+c)(b+d)=(b-c)^2$

(iv) $a:d=$ triplicate ratio of $(a-b):(b-c)$

(v) $\left(\frac{a-b}{c}+\frac{a-c}{b}\right)^2-\left(\frac{d-b}{c}+\frac{d-c}{b}\right)^2$

$=(a-d)^2\left(\frac{1}{c^2}-\frac{1}{b^2}\right)$

Solution:

a, b, c, d are in continued proportion

$\therefore \frac{a}{b}=\frac{b}{c}=\frac{c}{d}=k(say)$

$\therefore c=dk,\ b=ck=dk\,.\,k=dk^2,$

$a=bk=dk^2\,.\,k=dk^3$

(i) L.H.S. $=\frac{a^3+b^3+c^3}{b^3+c^3+d^3}$

$=\frac{(dk^3)^3+(dk^2)^3+(dk)^3}{(dk^2)^3+(dk)^3+d^3}$

$=\frac{d^3k^9+d^3k^6+d^3k^3}{d^3k^6+d^3k^3+d^3}$

$=\frac{d^3k^3(k^6+k^3+1)}{d^3(k^6+k^3+1)}=k^3$

R.H.S. $=\frac{a}{d}=\frac{dk^3}{d}=k^3$

$\therefore$ L.H.S. = R.H.S.

(ii) L.H.S. $= (a^2 - b^2)(c^2 - d^2)$

$= [(dk^3)^2 - (dk^2)^2]\,[(dk)^2 - d^2]$

$= (d^2k^6 - d^2k^4)(d^2k^2 - d^2)$

$= d^2k^4(k^2 - 1)\,d^2(k^2 - 1) = d^4k^4(k^2 - 1)^2$

R.H.S. $= (b^2 - c^2)^2 = [(dk^2)^2 - (dk)^2]^2$

$= [d^2k^4 - d^2k^2]^2 = [d^2k^2(k^2 - 1)]^2$

$= d^4k^4(k^2 - 1)^2$

$\therefore$ L.H.S. = R.H.S.

(iii) L.H.S. $= (a + d)(b + c) - (a + c)(b + d)$

$= (dk^3 + d)(dk^2 + dk) - (dk^3 + dk)(dk^2 + d)$

$= d(k^3 + 1)\,dk(k + 1) - dk(k^2 + 1)\,d(k^2 + 1)$

$= d^2k(k + 1)(k^3 + 1) - d^2k(k^2 + 1)(k^2 + 1)$

$= d^2k[k^4 + k^3 + k + 1 - k^4 - 2k^2 - 1]$

$= d^2k[k^3 - 2k^2 + k] = d^2k^2[k^2 - 2k + 1]$

$= d^2k^2(k - 1)^2$

R.H.S. $= (b - c)^2 = (dk^2 - dk)^2 = d^2k^2(k - 1)^2$

$\therefore$ L.H.S. = R.H.S.

(*iv*) $a : d$ = triplicate ratio of $(a-b) : (b-c)$

$= (a-b)^3 : (b-c)^3$

L.H.S. $= a : d = \dfrac{a}{d} = \dfrac{dk^3}{d} = k^3$

R.H.S. $= \dfrac{(a-b)^3}{(b-c)^3}$

$$= \frac{(dk^3 - dk^2)^3}{(dk^2 - dk)^3} = \frac{d^3k^6(k-1)^3}{d^3k^3(k-1)^3} = k^3$$

$\therefore$ L.H.S. = R.H.S.

(*v*) L.H.S. =

$$\left(\frac{a-b}{c} + \frac{a-c}{b}\right)^2 - \left(\frac{d-b}{c} + \frac{d-c}{b}\right)^2$$

$$= \left(\frac{dk^3 - dk^2}{dk} + \frac{dk^3 - dk}{dk^2}\right)^2$$

$$- \left(\frac{d-dk^2}{dk} + \frac{d-dk}{dk^2}\right)^2$$

$$= \left(\frac{dk^2(k-1)}{dk} + \frac{dk(k^2-1)}{dk^2}\right)^2$$

$$-\left(\frac{d(1-k^2)}{dk}+\frac{d(1-k)}{dk^2}\right)^2$$

$$=\left(k(k-1)+\frac{k^2-1}{k}\right)^2-\left(\frac{1-k^2}{k}+\frac{1-k}{k^2}\right)^2$$

$$=\left(\frac{k^2(k-1)+(k^2-1)}{k}\right)^2-\left(\frac{k(1-k^2)+1-k}{k^2}\right)^2$$

$$=\left(\frac{k^3-k^2+k^2-1}{k}\right)^2-\left(\frac{k-k^3+1-k}{k^2}\right)^2$$

$$=\frac{(k^3-1)^2}{k^2}-\frac{(-k^3+1)^2}{k^4}$$

$$=\frac{(k^3-1)^2}{k^2}-\frac{(1-k^3)^2}{k^4}$$

$$=\left(\frac{(k^3-1)}{k^2}\right)^2\left(1-\frac{1}{k^2}\right)=\frac{(k^3-1)^2(k^2-1)}{k^4}$$

$$=\frac{(k^3-1)^2(k^2-1)}{k^4}.$$

$$\text{R.H.S.}=(a-d)^2\left(\frac{1}{c^2}-\frac{1}{b^2}\right)$$

$$=(dk^3-d)^2\left(\frac{1}{d^2k^2}-\frac{1}{d^2k^4}\right)$$

$$=d^2(k^3-1)^2\left(\frac{k^2-1}{d^2k^4}\right)=\frac{(k^3-1)^2(k^2-1)}{k^4}$$

$\therefore$ L.H.S. = R.H.S.